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OPTIMAL ELASTIC DOMAINS OF MAXIMUM STIFFNESS*

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The problem of maximizing the stiffness (minimizing the work of the external forces) of an elastic domain of given volume is examined. Control is achieved by the shape of the domain /1-3/. A necessary Legendre condition is obtained as a result of investigating the second variation. The optimal solution can be found in the class of multiconnected elastic domains. The problem of increasing the connectedness is solved by using the Weierstrass necessary condition for a strong maximum. A dual problem is constructed to estimate the global maximum. An example is presented of the domain of maximum stiffness.

1. Formulation of the problem. Concepts of a design domain, an allowable domain, and a variation domain were introduced in /4/ and existence theorems were proved for the first and second variations of the displacements of an elastic domain. Let the set of allowable domains be denoted by Ω for which

$$\operatorname{mes}\Omega = \theta < \operatorname{mes}\Omega^\circ \tag{1.1}$$

where Ω° is the design domain in terms of $O^s(\lambda)$ (here $0<\lambda<1$ and s is an integer characterising the smoothness of the boundary Γ of the domain Ω /4/).

Let us formulate the optimum design problem. Suppose we are given the shear modulus μ, Poisson's ratio ν, the design domain Ω°, the coefficient θ satisfying the inequality (1.1), the external load factor $\mathbf{F}$ acting on the boundary Γ_F°, and the section of the boundary Γ_u° on which the displacements of the elastic domain equal zero. It is required to find

$$\inf J(\mathbf{u}),\quad J=\int_{\Gamma_F} F_i u_i\,d\Gamma,\quad \forall\Omega\in O^s(\lambda) \tag{1.2}$$

where $\mathbf{u}=u_i\mathbf{e}_i$ is the solution of the integral identity

$$\int_\Omega A(\mathbf{u},\mathbf{v})\,dx-\int_{\Gamma_F} F_i v_i\,d\Gamma=0,\quad \forall\mathbf{v}\in V(\Omega) \tag{1.3}$$

$$V(\Omega)=\{\mathbf{v}=v_i(\mathbf{x})\,\mathbf{e}_i\,|\,v_i\in W_2^{(1)}(\Omega),\ v_i(\mathbf{y})=0,\ \mathbf{y}\in\Gamma_u\}$$

$$\mathbf{x}=x_i\mathbf{e}_i,\quad A(\mathbf{u},\mathbf{v})=a_{ijkl}\varepsilon_{ij}(\mathbf{u})\,\varepsilon_{kl}(\mathbf{v})$$

$$\varepsilon_{kl}(\mathbf{v})={}^1/_2\,(\partial v_k/\partial x_l+\partial v_l/\partial x_k)$$

$\mathbf{e}_i$ are the unit vectors of the Cartesian coordinate system, x_i are Cartesian coordinates, u_i are displacements of the elastic domain, $A(\mathbf{v},\mathbf{v})$ is twice the specific elastic strain potential energy, and $W_2^{(1)}(\Omega)$ is the space of Sobolev functions /5/. Here, and everywhere henceforth summation is assumed to be between 1 and N, where $N=2$ or 3, over the repeated subscripts i,j,k,l,m,n in the products.

2. First and second variations. Let us form the expanded functional for which we append the left-hand side of (1.3) to the right-hand side of (1.2) and we find the first variation

$$\delta J=\int_{\Omega^*} A(\delta\mathbf{u},\mathbf{v})\,dx+\int_{\Gamma_F} F_i\delta u_i\,d\Gamma+\int_{\Gamma^*} A(\mathbf{u}^*,\mathbf{v})\,\delta r\,d\Gamma. \tag{2.1}$$

where δr and $\delta\mathbf{u}$ are variations of the domain Ω^* and the displacement $\mathbf{u}^*$. It is assumed here that the optimal solution is $\Omega^*\in O^s(\lambda),\ 0<\lambda<1,\ s\geqslant 4$. We set $\mathbf{v}=-\mathbf{u}^*$. We then obtain the inequality

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$$\delta J = -\int_{\Gamma^*} A(\mathbf{u}^*, \mathbf{u}^*)\, \delta r\, d\Gamma \geqslant 0 \tag{2.2}$$

from (1.3) and the necessary condition for a minimum $\delta J \geqslant 0$, which is valid for any allowable δr /4/ satisfying the condition

$$\int_{\Gamma^*} \delta r\, d\Gamma = 0 \tag{2.3}$$

The following obvious theorem is obtained from inequlity (2.2) and Eq.(2.3).

Theorem 1. Let $\Omega^* \in O^s(\lambda)$ where $s \geqslant 4,\ 0 < \lambda < 1$ and the vector function $\mathbf{u}^*$, which is a solution of the integral identity (1.3) for Ω^* give a minimum to the functional J. Then a $\zeta > 0$ is found such that

$$A(\mathbf{u}^*(\mathbf{y}),\ \mathbf{u}^*(\mathbf{y})) = \zeta,\ \ \mathbf{y} \in \Gamma^* \diagdown \Gamma^\circ \tag{2.4}$$
$$A(\mathbf{u}^*(\mathbf{y}),\ \mathbf{u}^*(\mathbf{y})) \geqslant \zeta,\ \ \mathbf{y} \in (\Gamma^* \cap \Gamma^\circ)\ \ (\Gamma_F{}^\circ \cup \Gamma_u{}^\circ)$$

We now obtain the second variation of the functional (1.2). Taking into account that $\mathbf{v} = -u^*$, we obtain

$$\delta^2 J = -\int_{\Gamma^*} \Big\{ 2A(\mathbf{u}^*, \delta\mathbf{u})\, \delta r + \frac{\partial A(\mathbf{u}^*, \mathbf{u}^*)}{\partial \tau_1} \delta^2 r + A(\mathbf{u}^*, \mathbf{u}^*)\, [\delta^2 r - I_1(\mathbf{t})\, \delta r^2] \Big\}\, d\Gamma \tag{2.5}$$

where $\delta^2 r$ is the second variation of the domain Ω^*, $I_1(\mathbf{t})$ is the first invariant of the curvature tensor $\mathbf{t}$ of the boundary Γ^* and τ_1 is a coordinate orthogonal to Γ^* /4/.

We shall examine the variations δr and $\delta^2 r$ on the sections $\Gamma^* \diagdown \Gamma^\circ$ of the boundary. In this case

$$\delta J = 0,\ \ \delta^2 J \geqslant 0 \tag{2.6}$$

$A(\mathbf{u}^*, \mathbf{u}^*) = \zeta$ by virtue of Theorem 1, while

$$\int_{\Gamma^*} [\delta^2 r - I_1(\mathbf{t})\, \delta r^2]\, d\Gamma = 0 \tag{2.7}$$

by virtue of the constraint (1.1). It follows from (2.5)-(2.7) that

$$-\int_{\Gamma^*} \Big[2A(\mathbf{u}^*, \delta\mathbf{u})\, \delta r + \frac{\partial A(\mathbf{u}^*, \mathbf{u}^*)}{\partial \tau_1} \delta r^2 \Big]\, d\Gamma \geqslant 0 \tag{2.8}$$

Transforming the first component in the integrand of (2.8), we find

$$A(\mathbf{u}^*, \delta\mathbf{u})\, \delta r = \overline{\nabla} \cdot (\sigma(\mathbf{u}^*) \cdot \delta\mathbf{u}\delta r) + \mathbf{r}_1 \cdot \frac{\partial \sigma(\mathbf{u}^*)}{\partial \tau_1} \cdot \delta\mathbf{u}\delta r + \mathbf{r}_1 \cdot \sigma(\mathbf{u}^*) \cdot \frac{\partial \delta\mathbf{u}}{\partial \tau_1} \delta r - \nabla \cdot \sigma(\mathbf{u}^*) \cdot \delta\mathbf{u}\delta r - \overline{\nabla}\delta r \cdot \sigma(\mathbf{u}^*) \cdot \delta\mathbf{u} \tag{2.9}$$

where $\mathbf{r}_1$ is the direction of the external normal to Γ^*, $\nabla = \overline{\nabla} + \mathbf{r}_1 \partial/\partial\tau_1$ is the Hamilton operator, and $\sigma_{ij}(\mathbf{u}^*) = a_{ijkl}\varepsilon_{kl}(\mathbf{u}^*)$ are stress tensor components. We note that the third component equals zero since $\mathbf{r}_1 \cdot \sigma(\mathbf{u}^*) = \mathbf{0}$ by virtue of the boundary conditions on the optimal domain boundary. Substituting the right-hand side of (2.9) into (2.8) and integrating the first component, we obtain

$$\int_{\Gamma^*} \Big[2\overline{\nabla} \cdot (\sigma(\mathbf{u}^*)\, \delta r) \cdot \delta\mathbf{u} - \frac{\partial A(\mathbf{u}^*, \mathbf{u}^*)}{\partial \tau_1} \delta r^2 \Big]\, d\Gamma \geqslant 0 \tag{2.10}$$

The first component in the integrand of (2.10) is a quadratic form dependent on δr and $\overline{\nabla}\delta r$, since $\delta\mathbf{u}$ is determined by the integral identity, the force load in which is $\overline{\nabla} \cdot [\sigma(\mathbf{u}^*)\, \delta r]$ /4/. Using it we obtain

$$\int_{\Gamma^*} \overline{\nabla} \cdot [\sigma(\mathbf{u}^*)\, \delta r] \cdot \delta\mathbf{u}\, d\Gamma = \int_{\Omega^*} A(\delta\mathbf{u}, \delta\mathbf{u})\, dx \geqslant 0$$

from which it follows that the Legendre necessary condition for the stiffness maximization problem is always satisfied.

3. The Weierstrass necessary condition. To obtain the Weierstrass necessary condition at the point $\mathbf{x}_0 \in \Omega^* \in O^s(\lambda)$, $0 < \lambda < 1$, we select an $\eta_0 > 0$, such that the sphere $\bar{U}(\mathbf{x}_0, \eta_0) \subset \Omega^*$, $(U(\mathbf{x}_0, \eta_0) = \{\mathbf{x} \| \mathbf{x} - \mathbf{x}_0 | < \eta_0\})$. Let us examine the simply-connected domain Ω_0 that is a star relative to $\mathbf{x}_0$, where, $\bar{\Omega}_0 \subset U(\mathbf{x}_0, \eta_0)$. We take any vector $\mathbf{y} \in \Gamma_0$ (Γ_0 is the boundary of Ω_0) and we draw the vector $\mathbf{r}(\mathbf{y})$ from the point $\mathbf{x}_0$ in it. If the set of points $\eta\mathbf{r}(\mathbf{y})$ is considered then the boundary $\Gamma_0(\eta)$ is obtained that separates the domain $\Omega_0(\eta)$, where $\Omega_0 = \Omega_0(1)$, $\Gamma_0 = \Gamma_0(1)$, $\bar{\Omega}_0(\eta) \subset U(\mathbf{x}_0, \eta_0)$, $0 < \eta < \eta_0$. The domain $\Omega_0(\eta)$ can be obtained from Ω_0 if all its linear dimensions change η-fold, consequently

$$\operatorname{mes}\Omega_0(\eta) = \eta^N \operatorname{mes}\Omega_0 \tag{3.1}$$

We now construct a family of domains $\Omega(\eta) \in O^s(\lambda)$. To do this we "cut out" a cavity $\bar{\Omega}_0(\eta)$, $0 < \eta \leqslant \eta_0$, in Ω^* and in a certain $(N-1)$-dimensional sector of the boundary $\Gamma^* \setminus \Gamma^\circ$ we give the generating boundary function $r(\mathbf{y}, \eta) \geqslant 0$ /4/. It follows from the construction of $\Omega(\eta)$ that

$$\delta r(\mathbf{y}) = \ldots = \delta^{N-1} r(\mathbf{y}) = 0, \quad \delta^N r(\mathbf{y}) \geqslant 0 \tag{3.2}$$

$$\int_{\Gamma^*} \delta^N r \, d\Gamma = \operatorname{mes}\Omega_0 N! \tag{3.3}$$

We continue $\mathbf{u}^*$ into the domain $\Omega(\eta) \setminus \bar{\Omega}^*$ so that $\mathbf{u}^*$ and all the first derivatives of u^* are continuous when going from Ω^* into $\Omega(\eta)$ in the $(N-1)$-dimensional sector in which $r(\mathbf{y}, \eta) \geqslant 0$ and we represent the expanded functional ($J(\mathbf{u})$ plus the left-hand side of the integral identity (1.3)) in the form

$$J = \int\limits_{U(\mathbf{x}_0, \eta) \setminus \bar{\Omega}_0(\eta)} A(\mathbf{u}, \mathbf{u}^*)\, dx - \int\limits_{\Omega^* \setminus \bar{U}(\mathbf{x}_0, \eta_0)} A(\mathbf{u}, \mathbf{u}^*)\, dx - \int\limits_{\Omega(\eta) \setminus \bar{\Omega}^*} A(\mathbf{u}, \mathbf{u}^*)\, dx + \int\limits_{\Gamma_F} F_i (u_i + u_i^*)\, d\Gamma$$

In the domain $U(\mathbf{x}_0, \eta_0) \setminus \bar{\Omega}_0(\eta)$ $\mathbf{u}^*$ is an infinitely differentiable function /6/, consequently, applying the formula

$$A(\mathbf{u}, \mathbf{u}^*) = \nabla \cdot (\boldsymbol{\sigma}(\mathbf{u}^*) \cdot \mathbf{u}) - [\nabla \cdot \boldsymbol{\sigma}(\mathbf{u}^*)] \cdot \mathbf{u}$$

in the first integral and using the Gauss's Theorem, we obtain

$$J = J_1 + J_2 \tag{3.4}$$

$$J_1 = -\int\limits_{\Gamma_0(\eta)} \mathbf{r}_1 \cdot \boldsymbol{\sigma}(\mathbf{u}^*) \cdot \mathbf{u}\, d\Gamma$$

$$J_2 = -\int\limits_{\Omega^* \setminus \bar{\Omega}(\eta)} A(\mathbf{u}, \mathbf{u}^*)\, dx + \int\limits_{\Gamma_F} F_i u_i^*\, d\Gamma \tag{3.5}$$

Let us find the variation J_2. Since the conditions (3.2) are satisfied for $r(\mathbf{y}, \eta)$, we have

$$\delta J_2 = \ldots = \delta^{N-1} J_2 = 0, \quad \delta^N J_2 = -\int\limits_{\Gamma^*} A(\mathbf{u}^*, \mathbf{u}^*)\, \delta^N r\, d\Gamma \tag{3.6}$$

from which it indeed follows from (2.4) and (3.3) that

$$\delta^N J_2 = -\zeta N! \operatorname{mes}\Omega_0 \tag{3.7}$$

Theorem 2. Let the conditions of the theorem be satisfied and $\delta J_1 = \ldots = \delta^{N-1} J_1 = 0$. Then the Weierstrass condition

$$\delta^N J_1 \geqslant \zeta N! \operatorname{mes}\Omega_0, \quad \forall \mathbf{x} \in \Omega^* \tag{3.8}$$

should be satisfied.

Proof. The domain Ω^* and $\mathbf{u}^*$ give a minimum to the functional J from which, together with the conditions of the theorem and (3.6), it follows that $\delta J = \ldots = \delta^{N-1} J = 0$, $\delta^N J \geqslant 0$. Taking (3.4) and (3.7) into account, we obtain (3.8).

4. The Weierstrass necessary condition for an elliptical hole. ($N = 2$). It is not possible to determine the left-hand side of the Weierstrass condition (3.8) for arbitrary holes Ω_0. However, in the two-dimensional case the solution $\mathbf{u}$ can be determined for certain hole shapes, elliptical, hypotrochoidal, and certain others, as $\eta \to 0$ and the inequality (3.8) can be expressed in terms of the stresses.

Let Ω_0 be an elliptical hole with major and minor semi-axes $a = \eta(1 + \xi)$, $b = \eta(1 - \xi)$, $0 \leqslant$

$\xi \leqslant 1$, whose centre is placed at the point x_0 (without loss of generality we can set $\mathbf{x}_0 = 0$). We will consider that the principal stress $\sigma_1 = \sigma_1(\mathbf{u}^*(0))$ of the tensor $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{u}^*(0))$ acts at an angle β to the ellipse semimajor axis. The solution $\mathbf{u}(\mathbf{x}, \eta)$ on the right-hand side of (3.5) is represented in the form of the sum $\mathbf{u} = \mathbf{u}^*(\mathbf{x}) + \bar{\mathbf{u}}(\mathbf{x}, \eta)$, where $\bar{\mathbf{u}}$ is the solution of the integral identity

$$\int\limits_{\Omega(\eta)} A(\bar{\mathbf{u}}, \mathbf{v})\, dx + \int\limits_{\Gamma_0(\eta)} \mathbf{r}_1 \cdot \boldsymbol{\sigma}(\mathbf{u}^*) \cdot \mathbf{v}\, d\Gamma = 0, \quad \forall \mathbf{v} \in V(\Omega(\eta)) \tag{4.1}$$

(Here $\mathbf{r}_1$ is the direction of the normal to $\Gamma_0(\eta)$ that is external with respect to $\Omega(\eta)$). It is known that as $\eta \to 0$ the solution $\bar{\mathbf{u}}$ in the neighbourhood of $\Gamma_0(\eta)$ is identical, to the accuracy η, with $\mathbf{u}^0(\mathbf{x}, \eta)$ — the solution for an infinite plane with an elliptical hole on whose boundary the load $-\mathbf{r}_1 \cdot \boldsymbol{\sigma}(\mathbf{u}^*)$ acts. This solution can be found by the Kolosov-Muskhelishvili formulas /7/. Since it is necessary to evaluate (3.5), we then present $\mathbf{u}^0$ on the boundary of the ellipse under consideration

$$\begin{aligned} &u_1^\circ + iu_2^\circ = \eta\,(4\mu)^{-1}[\sigma_1 \Lambda(\beta) + \sigma_2 \Lambda(-\beta)] \\ &\Lambda(\beta) = \varkappa e^{-i\theta}(e^{2i\beta} - \xi) + e^{i\theta}(1 - \xi e^{2i\beta}) \end{aligned} \tag{4.2}$$

Here θ is the angle measured from the x_1 axis while $\varkappa = 3 - 4\nu$ for a plane state of strain and $\varkappa = (3 - \nu)(1 + \nu)^{-1}$ for a plane state of stress. Substituting $\mathbf{u}^*$, (4.2), $\mathbf{r}_1 = -[(1 - \xi)\cos\theta \mathbf{e}_1 + (1 + \xi)\sin\theta \mathbf{e}_2]/R$ and $d\Gamma = \eta R\, d\theta$ into the right-hand side of (3.5), where $R = \sqrt{1 - 2\xi\cos 2\theta + \xi^2}$ we obtain, apart from components proportional to η^2, after reduction

$$\begin{aligned} &J_1 \approx \pi\eta^2(1 - \xi^2)\, A(\mathbf{u}^*, \mathbf{u}^*) + \pi\eta^2(4\mu)^{-1}\psi(\beta, \xi) \\ &\psi(\beta, \xi) = (1 + \xi^2\varkappa)(\sigma_1 + \sigma_2)^2 + (\varkappa + \xi^2)(\sigma_1 - \sigma_2)^2 - \\ &\quad 2\xi(1 + \varkappa)(\sigma_1^2 - \sigma_2^2)\cos 2\beta \end{aligned} \tag{4.3}$$

For the ellipse $\operatorname{mes}\Omega_0 = \pi(1 - \xi^2)$. Substituting this expression and (4.3) into (3.8), we obtain the Weierstrass necessary condition for an ellipse

$$\psi(\beta, \xi)\,[4\mu(1 - \xi^2)]^{-1} \geqslant \zeta - A(\mathbf{u}^*, \mathbf{u}^*) \tag{4.4}$$

It should be satisfied for any allowable values β, ξ consequently, the problem

$$\min \psi(\beta, \xi)\,[4\mu(1 - \xi^2)]^{-1}, \; 0 \leqslant \beta \leqslant \pi, \; 0 \leqslant \xi \leqslant 1 \tag{4.5}$$

should be solved.

To be specific we set $\sigma_1^2 \geqslant \sigma_2^2$. Then the solution of problem (4.5) is obtained for

$$\beta_0 = 0, \quad \xi_0 = \begin{cases} (\sigma_1 + \sigma_2)(\sigma_1 - \sigma_2)^{-1}, & -1 \leqslant \sigma_2\sigma_1^{-1} \leqslant 0 \\ (\sigma_1 - \sigma_2)(\sigma_1 + \sigma_2)^{-1}, & 0 \leqslant \sigma_2\sigma_1^{-1} \leqslant 1 \end{cases}$$

$$\frac{\psi(\xi_0, \beta_0)}{4\mu(1 - \xi_0^2)} = \begin{cases} -4\varkappa\sigma_1\sigma_2, & -1 \leqslant \sigma_2\sigma_1^{-1} \leqslant 0 \\ 4\sigma_1\sigma_2, & 0 \leqslant \sigma_2\sigma_1^{-1} \leqslant 1 \end{cases}$$

and the Weierstrass necessary condition takes the form

$$\begin{aligned} &A(\mathbf{u}^*, \mathbf{u}^*) - \varkappa\sigma_1\sigma_2\mu^{-1} \geqslant \zeta, \; -1 \leqslant \sigma_2\sigma_1^{-1} \leqslant 0 \\ &A(\mathbf{u}^*, \mathbf{u}^*) + \sigma_1\sigma_2\mu^{-1} \geqslant \zeta, \; 0 \leqslant \sigma_2\sigma_1^{-1} \leqslant 1 \\ &\mathbf{u}^* = \mathbf{u}^*(\mathbf{x}_0), \; \sigma_k = \sigma_k(\mathbf{u}^*(\mathbf{x}_0)) \end{aligned} \tag{4.6}$$

5. Weierstrass necessary condition for an ellipsoidal cavity ($N = 3$).

Let Ω_0 be an ellipsoidal cavity with semi-axes $\eta a_1 \geqslant \eta a_2 \geqslant \eta a_3$ whose centre is placed at the point $\mathbf{x}_0$ ($\mathbf{x}_0$ is set equal to 0). The solution $\mathbf{u}(\mathbf{x}, \eta)$ on the right-hand side of (3.5) is represented in the form of the sum $u = \mathbf{u}^*(\mathbf{x}) + \bar{\mathbf{u}}(\mathbf{x}, \eta)$, where $\bar{\mathbf{u}}$ is the solution of the integral identity (4.1). Exactly as in Sect.4, we find $\mathbf{u}^0(\mathbf{x}, \eta)$ - the solution of the problem of the three-dimensional space with an ellipsoidal cavity on whose surface the load $-\mathbf{r}_1 \cdot \boldsymbol{\sigma}(\mathbf{u}^*)$ acts. This solution is known /8/. We present it at points of the boundary Γ_0

$$\begin{aligned} &u_i^\circ = W_{ij}x_j(2\mu)^{-1}, \; W_{ii} = 2Z_{ik}B_{kk} \\ &W_{ij} = \left[4(1 - \nu)\omega_j + 2\frac{\rho^2 - \xi_i}{\xi_i - \xi_j}\omega_i - 2\frac{\rho^2 - \xi_j}{\xi_i - \xi_j}\omega_j\right] B_{ij}, \quad (i \neq j) \end{aligned} \tag{5.1}$$

$$Z_{i1}=(1-2\nu)\,\omega_i,\quad Z_{i2}=(1-2\nu)\,\omega_i+(2\rho\Delta)^{-1}$$
$$Z_{i3}=(1-2\nu)\frac{(c_1-c_2)(\rho^2-\xi_i)}{(c_1-\xi_i)(c_2-\xi_i)}\,\omega_i-\frac{(\rho^2-c_1)^2}{c_1-\xi_i}\,\omega_4+\frac{(\rho^2-c_2)^2}{c_2-\xi_i}\,\omega_5$$
$$\xi_1=0,\quad \xi_2=e^2,\quad \xi_3=1,\quad a_k=\sqrt{\rho^2-\xi_k}$$
$$\Delta=\Delta(\rho)=\sqrt{(\rho^2-e^2)(\rho^2-1)},\quad c_{1,2}=(1+e^2\pm\sqrt{1+e^2+e^4})/3$$
$$\omega_k=\int_\rho^\infty\frac{d\lambda}{(\lambda^2-\xi_k)\,\Delta(\lambda)},\quad \omega_{3+q}=\int_\rho^\infty\frac{d\lambda}{(\lambda^2-c_q)\,\Delta(\lambda)},\quad q=1,2$$

The constants B_{ij} are determined by the relationships

$$B_{ij}=\sigma_{ij}(2D_{ij})^{-1},\quad i\neq j,\quad B_{ii}=-Q_{ik}\sigma_{kk}/2$$
$$D_{ij}=(1-\nu)\,[(\rho\Delta)^{-1}-\omega_i-\omega_j]-[(\rho^2-\xi_i)\,\omega_i-(\rho^2-\xi_j)\,\omega_j]\,(\xi_i-\xi_j)^{-1}$$

where Q_{ij} are elements of the matrix P^{-1}, and the elements of the matrix P are determined by the equalities

$$P_{q1}=(1-2\nu)\,[\omega_q-(\rho\Delta)^{-1}],\quad P_{q2}=(1-2\nu)\,\omega_q(\rho^2-\xi_q)^{-1}+\frac{\nu\omega_k}{\rho^2-\xi_k}-\frac{1}{2\rho\Delta}\left(\frac{1-2\nu}{\rho^2-\xi_q}-\sum_{k=1}^{3}\frac{1}{\rho^2-\xi_k}\right)$$
$$P_{q3}=(1-2\nu)\frac{(c_1-c_2)(\rho^2-\xi_q)}{(c_1-\xi_q)(c_2-\xi_q)}\,\omega_q+\nu\frac{(c_1-c_2)(\rho^2-\xi_k)}{(c_1-\xi_k)(c_2-\xi_k)}\,\omega_k-\frac{(1-\nu)(c_1-c_2)(\rho^2-\xi_k)}{(c_1-\xi_q)(c_2-\xi_q)\,\rho\Delta}-\frac{(\rho^2-c_1)^2}{c_1-\xi_q}\,\omega_4+\frac{(\rho^2-c_2)^2}{c_2-\xi_q}\,\omega_5$$
$$\Delta=\Delta(\rho),\quad q=1,2,3$$

Substituting (5.1) and $\mathbf{u}^*$ into the right-hand side of (3.5), we obtain

$$J_1\approx\frac{4\pi\eta^3}{3}\,a_1a_2a_3\left[A(\mathbf{u}^*,\mathbf{u}^*)+\frac{1}{2\mu}\,\psi\right] \tag{5.2}$$
$$\psi=-Z_{ik}Q_{kj}\sigma_{ii}\sigma_{jj}+\sum_{p,q=1,\ p\neq q}^{3}\sigma_{pq}^2\,[(1-\nu)(D_{pq}\rho\Delta)^{-1}-1]$$

For an ellipsoid $\operatorname{mes}\Omega_0=4\pi a_1a_2a_3/3$. Substituting this expression and (5.2) into (3.8), we obtain the Weierstrass necessary condition for an ellipsoid

$$\psi\,(2\mu)^{-1}\geqslant\zeta-A\,(\mathbf{u}^*,\ \mathbf{u}^*) \tag{5.3}$$

Inequality (5.3) should be satisfied for any allowable ρ, e and for any ellipsoid location relative to the principal axes of the tensor $\sigma(\mathbf{u}^*(\mathbf{x}_0))$. Let γ denote the matrix of the direction cosines between the principal axes of the tensor $\sigma(\mathbf{u}^*(\mathbf{x}_0))$ and the directions $\mathbf{e}_i$ of the Cartesian coordinate system (γ_{ij} is the cosine of the angle between the direction $\mathbf{e}_i$ and the direction $\sigma_j(\mathbf{u}^*(\mathbf{x}_0))$). We introduce the angle of precession β_1, the nutation β_2, and the pure rotation β_3 (the Euler angles), we express γ_{ij} in terms of β_k (see /9/), we represent the components of the tensor $\sigma(\mathbf{u}^*(\mathbf{x}_0))$ in the form $\sigma_{ij}=\sigma_k\gamma_{ik}\gamma_{jk}$ and we substitute them into the left-hand side of (5.3). Then the function ψ will depend on the parameters ρ, e of the ellipsoid and the angles β_1, β_2, β_3 that is

$$\psi\,(\rho,\ e,\ \beta_1,\ \beta_2,\ \beta_3)\,(2\mu)^{-1}\geqslant\zeta-A\,(\mathbf{u}^*(\mathbf{x}_0),\ \mathbf{u}^*(\mathbf{x}_0)) \tag{5.4}$$

Inequality (5.4) should be satisfied for any allowable ρ, e, β_k, and consequently it is necessary to solve the problem

$$\psi^*=\min\psi\,(\rho,\ e,\ \beta_1,\ \beta_2,\ \beta_3),\quad 1\leqslant\rho<\infty \tag{5.5}$$
$$0\leqslant e\leqslant 1,\quad 0\leqslant\beta_k\leqslant\pi$$

Since an explicit expression for ψ with respect to the arguments is not obtained successfully, problem (5.5) can only be solved numerically. To be specific, we set $\sigma_1^2\geqslant\sigma_2^2$, $\sigma_1^2\geqslant\sigma_3^2$ and use the notation $\alpha_1=\sigma_2/\sigma_1$, $\alpha_2=\sigma_3/\sigma_1$. The relationship ψ^*/σ_1^2 is presented in Fig.1 for different values of α_1 and α_2 for $\nu=0.3$.

6. Dual estimate. To obtain the dual estimate we consider the bilinear form

$$M(\mathbf{u},\mathbf{v})=\int_\Omega[A(\mathbf{u},\mathbf{u})+A(\mathbf{u},\mathbf{v})]\,dx-\int_{\Gamma_F}F_iv_i\,d\Gamma \tag{6.1}$$

$$\forall\ \Omega \in O^s(\lambda),\quad \forall \mathbf{u},\quad \mathbf{v} \in V(\Omega)$$

The functional

$$M^\circ(\mathbf{u}) = \sup M(\mathbf{u},\ \mathbf{v}),\quad \forall \mathbf{v} \in V(\Omega)$$

obviously equals the work of the external forces and takes finite values for $\mathbf{u}$ that satisfy the integral identity (1.3).

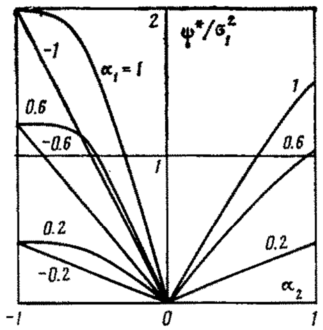

Fig.1

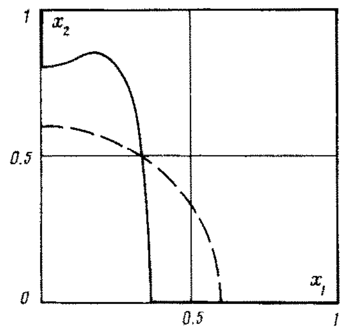

Fig.2

We now construct the functional

$$M_0(\mathbf{v}) = \inf M(\mathbf{u},\ \mathbf{v}),\quad \forall \Omega \in O,\quad \forall \varepsilon_{ij}(\mathbf{u}) \in L_2(\Omega)$$

Since the integrand in the first integral on the right-hand side of (6.1) is a strongly convex function of $\varepsilon_{ij}(\mathbf{u})$, the necessary and sufficient condition $\inf M(\mathbf{u},\ \mathbf{v})$, $\forall \varepsilon_{ij}(\mathbf{u}) \in L_2(\Omega)$ is the equality $\varepsilon(\mathbf{u}) = -\varepsilon(\mathbf{v})/2$, from which $\mathbf{u} = -\mathbf{v}/2$, $\forall \mathbf{x} \in \Omega$ follows while

$$\inf_{\varepsilon_{ij}(\mathbf{u}) \in L_2(\Omega)} M(\mathbf{u}, \mathbf{v}) = -\frac{1}{4}\int_\Omega A(\mathbf{v}, \mathbf{v})\,dx - \int_{\Gamma_F} F_i v_i\, d\Gamma$$

We introduce the constant ζ for each fixed $\mathbf{v} \in V(\Omega^\circ)$ such that

$$\begin{aligned} \Omega_v &= \{\mathbf{x} \in \Omega^\circ \mid A(\mathbf{v},\ \mathbf{v}) \geqslant \zeta\},\quad \operatorname{mes} \Omega_v = \theta \\ \Omega^\circ \setminus \Omega_v &= \{\mathbf{x} \in \Omega^\circ \mid A(\mathbf{v},\ \mathbf{v}) \leqslant \zeta\} \end{aligned} \tag{6.2}$$

Now it is easy to find the functional

$$M_0(\mathbf{v}) = -\frac{1}{4}\int_{\Omega_v} A(\mathbf{v},\ \mathbf{v})dx - \int_{\Gamma_F} F_i v_i\, d\Gamma,\ \forall \mathbf{v} \in V(\Omega^\circ) \tag{6.3}$$

for the domain Ω_v that satisfies conditions (6.2). Let us formulate the dual problem

$$\sup M_0(\mathbf{v}),\quad \forall \mathbf{v} \in V(\Omega^\circ)$$

and present the inequalities

$$\begin{aligned} &\sup{}^\circ M_0(\mathbf{v}) = \sup{}^\circ \inf{}^\circ M(\mathbf{u},\ \mathbf{v}) \leqslant \sup{}^\circ \inf M(\mathbf{u},\ \mathbf{v}) \leqslant \\ &\sup \inf M(\mathbf{u},\ \mathbf{v}) \leqslant \inf \sup M(\mathbf{u},\ \mathbf{v}) = \inf M^\circ(\mathbf{u}) \end{aligned} \tag{6.4}$$

Here $\sup^\circ$ denotes the operation $\sup$ for all $\mathbf{v} \in V(\Omega^\circ)$, $\inf^\circ$ is the operation $\inf$ for all $\Omega \in 0$, $\varepsilon_{ij}(\mathbf{u}) \in L_2(\Omega)$, the operations $\sup$ and $\inf$ are performed, respectively, for all $\mathbf{v} \in V(\Omega)$ and $\Omega \in O^s(\lambda)$, $\mathbf{u} \in V(\Omega)$; O is the set of measurable sets $\Omega \subset \Omega^\circ$, for which $\operatorname{mes} \Omega = \theta$.

Theorem 3. Let there exist the domain $\Omega^* \in O^s(\lambda)$, $0 < \lambda < 1$ and $\mathbf{u}^* \in V(\Omega^*)$, which is a solution of the integral identity (1.3) whose continuation on Ω° satisfies the condition

$$\begin{aligned} &A(\mathbf{u}^*,\ \mathbf{u}^*) \geqslant \zeta,\quad \forall \mathbf{x} \in \Omega^*,\quad A(\mathbf{u}^*,\ \mathbf{u}^*) \leqslant \zeta, \\ &\forall \mathbf{x} \in \Omega^\circ \setminus \Omega^* \end{aligned} \tag{6.5}$$

Then Ω^*, $\mathbf{u}^*$ are the optimal solution fo problem (1.2).

Proof. We set $\mathbf{v} = -2\mathbf{u}$. Since condition (6.5) is satisfied for Ω^*, then we can set $\Omega_v = \Omega^*$. It then follows from (6.3) that

$$M_0(-2\mathbf{u}^*) = -\int_{\Omega^*} A(\mathbf{u}^*, \mathbf{u}^*)\,dx + 2\int_{\Gamma_F} F_i u_i^*\, d\Gamma$$

But $\mathbf{u}^*$ satisfies the integral identity (1.3), so consequently

$$M_0(-2\mathbf{u}^*) = \int_{\Gamma_F} F_i u_i^* \, d\Gamma$$

from which, and inequality (6.4), it follows that the dual estimate and the value of the functional in problem (1.2) are identical. Therefore, Ω^*, $\mathbf{u}^*$ are the solution of problem (1.2).

Example. Let Ω^0 be a square with sides $2d$ on two edges of which a uniformly distributed load $\mathbf{F} = F\mathbf{e}_2$ acts. We will assume that a plane state of strain is realized.

As initial domain we take a square out of which a circle of radius g has been cut such that $4d^2 - \pi g^2 = \theta$, where θ is a given number. Starting from this initial approximation, using the necessary condition (2.4) as well as the scheme of partition into finite elements /2/, we obtain a domain of large stiffness.

A quadrant of the domain obtained as a result of optimization for $F = 1$ N/m^2, $d = 1$ m, $\mu = 7.7 \times 10^{10}$ N/m^2, $\nu = 0.3$, $\theta = 2.87$, and $g = 0.6$, is represented by the heavy one in Fig.2. The initial domain is represented by the dashed line. The work of the external forces equals $J_0 \simeq 0.635 \times 10^{-10}$ N and $J_1 \simeq 0.445 \times 10^{-10}$ N, respectively, for the initial and the improved domains, while the gain in stiffness is $\delta_1 = 1 - J_1/J_0 \simeq 0.299$.

We now obtain the dual estimate for $v_1 = -\alpha\nu x_1$, $v_2 = \alpha(1-\nu)x_2$. Substituting these functions into (6.3) and maximizing the value of M_0 in α, we obtain

$$J_* = \max_{\alpha} M_0(v) = 8d^4F^2(1-\nu)(\mu\theta)^{-1}$$

For the data used in the problem $J_* \simeq 0.254 \times 10^{-10}$ N, therefore, the greatest possible gain equals $\delta_* = 1 - J_*/J_0 \simeq 0.6$.

A further increase in the domain stiffness can be achieved because of an increase in its connectedness. Analysis shows that the Weierstrass condition (4.5) is not satisfied at almost all points of the domain.

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